The "square root" of the Dirac equation and solutions on superspace. [†]

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Abstract

The "square root" of the Dirac operator derived on the superspace is used to construct supersymmetric field equations. In addition to the recently found solution - a vector supermultiplet - it is demonstrated how other supermultiplets follow as solutions: a set of chiral and antichiral superfields, containing two spin 0 and two spin 1/2 component fields and a another set of two spin 3/2 and two real, spin 1 component fields. These supermultiplets are shown to obey appropriate (massless) equations of motion. The "square root" of the Dirac equation yields thus a complete set of fields and their equations necessary to construct renormalizable supersymmetric theories. The problem of masses and interaction is also discussed.

1. Introduction.

The idea to take the "square root" of the Dirac operator follows directly from the analogous procedure performed by Dirac on the Klein-Gordon operator [1]. Whereas in the first case the motivation was to linearize the operator in space-time derivatives, the form of the supersymmetry algebra [2] suggested that repeating this procedure would lead to the operator linear in supersymmetry generators or equivalently - spinorial derivatives. Such construction was presented some time ago [3] together with a set of supersymmetric field equations which result when acting with the "square root" operator on superfields.

To recall this construction in short let me write the Dirac equation in two-component notation and chiral representation:

$$-\left(\begin{array}{cc} i\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}\partial_\mu & m \\ m & i\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu \end{array} \right)\left(\begin{array}{c} \varphi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{array} \right) \equiv \mathcal{D}\left(\begin{array}{c} \varphi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{array} \right) = 0. \tag{1}$$

The Lorenz indices are denoted here by μ, ν, λ and ρ, the spinor indices by α and β.

I am looking for the operator $\mathcal{S}$ satysfying:

$$\mathcal{S}^\dagger\mathcal{S} = \mathcal{D}. \tag{2}$$

The solution proposed in Ref. [3] is:

$$\mathcal{S} = \frac{1}{\sqrt{2}}\left(\begin{array}{cc} D^\alpha & -\bar{D}_{\dot{\alpha}} \\ \bar{D}^{\dot{\alpha}} & D_\alpha \end{array} \right) \tag{3}$$

where the spinorial derivatives are defined on the superspace as:

$$\begin{array}{rcl} D_\alpha &=& \partial/\partial\theta^\alpha + i\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \\ \bar{D}_{\dot{\alpha}} &=& -\partial/\partial\bar{\theta}^{\dot{\alpha}} - i\theta^\alpha\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu. \end{array} \tag{4}$$

Indeed, using the anticomutation relations:

$$\begin{array}{rcl} \{D_\alpha, D_\beta\} &=& \left\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\right\} = 0, \\ \left\{D_\alpha, \bar{D}_{\dot{\beta}}\right\} &=& -2i\sigma^\mu{}_{\alpha\dot{\beta}}\partial_\mu \end{array} \tag{5}$$

we get

$$\mathcal{S}^\dagger\mathcal{S} = -\left(\begin{array}{cc} i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu & M \\ M & i\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu \end{array} \right) \tag{6}$$

where a scalar, hermitian operator

$$M = -\frac{1}{4}\left(DD + \bar{D}\bar{D}\right) \tag{7}$$

appears instead of the mass m. The operator $\mathcal{S}$ is thus the solution to our problem on the space of superfields Λ which satisfy

$$M \Lambda = m \Lambda. \tag{8}$$

Note another solution to the problem:

$$\mathcal{S}' = \frac{1}{\sqrt{2}}\left(\begin{array}{cc} \bar{D}^{\dot{\alpha}} & D_\alpha \\ -D^\alpha & \bar{D}_{\dot{\alpha}} \end{array}\right). \tag{9}$$

Acting with the operator $\mathcal{S}$ (or $\mathcal{S}'$) on a superfield we are able to construct a free field equation - the "square root" of the Dirac equation. The simplest 2 choices of superfields are [3]:

$$F = \left(\begin{array}{c} W_\alpha \\ \mathcal{H}^{\dot{\alpha}} \end{array}\right). \tag{10}$$

and

$$B = \left(\begin{array}{c} \Phi \\ V^{\alpha\dot{\alpha}} \end{array}\right) \tag{11}$$

leading to the equations:

$$\mathcal{S}F = 0, \quad \mathcal{S}B = 0,. \tag{12}$$

One can easily verify that the operator $\mathcal{S}'$ leads to the equivalent set of field equations. It is also obvious that due to Eq. (2) both superfields F and B satisfy the Dirac equation

$$\mathcal{D}F = 0, \quad \mathcal{D}B = 0. \tag{13}$$

Recently the equations:

$$\mathcal{S}F = 0 \tag{14}$$

(together with the condition $MF = mF$) were studied and solved in Ref. [4]. In the simplest case when $W_\alpha = \mathcal{H}_\alpha$, the solution was found to be the Maxwell supermultiplet:

$$\begin{aligned} W_\alpha = & - i\lambda_\alpha(y) + \left[\delta_\alpha{}^\beta d(y) - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta(\partial_\mu w_\nu(y) - \partial_\nu w_\mu(y))\right]\theta_\beta \\ & + \theta\theta\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}}(y) \end{aligned} \tag{15}$$

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with $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. The massless component fields $w_\mu(x)$ and $\lambda_\alpha(x)$ satisfy the Maxwell and Dirac equations respectively and $d = const.$

2. The equations and their solutions.

In this paper I study the other set of equations (12):

$$\mathcal{S}B = 0, \tag{16}$$

the superfield B satisfying in addition the condition (8): In terms of (in general complex) component superfields the equations read:

$$\begin{aligned} D^\alpha\Phi - \bar{D}_{\dot{\alpha}}V^{\alpha\dot{\alpha}} &= 0, \\ \bar{D}^{\dot{\alpha}}\Phi + D_\alpha V^{\alpha\dot{\alpha}} &= 0 \end{aligned} \tag{17}$$

and

$$\begin{aligned} M\Phi &= m\Phi, \\ MV^{\alpha\dot{\alpha}} &= mV^{\alpha\dot{\alpha}}. \end{aligned} \tag{18}$$

Multiplying the first Eq. (17) by D_α and making use of the second Eq. (17) I obtain

$$M\Phi = m\Phi = 0. \tag{19}$$

Out of two possibilities suppose first $\Phi = 0$ and the mass m arbitrary. The Eqs (17) simplify and it is easy to show in this case that

$$D^2 V^{\alpha\dot{\alpha}} = \bar{D}^2 V^{\alpha\dot{\alpha}} = 0, \tag{20}$$

which implies:

$$MV^{\alpha\dot{\alpha}} = mV^{\alpha\dot{\alpha}} = 0. \tag{21}$$

If we are interested in non-zero superfields, the mass m has to vanish, $m = 0$. The other possibility, $m = 0$ and Φ arbitrary, leads to similar conclusion. Indeed, a new constraint on Φ can be obtained by acting with the anticommutator $\left\{D_\alpha, \bar{D}_{\dot{\beta}}\right\}$ on the superfield Φ and using Eqs. (17):

$$\left\{D_\alpha, \bar{D}_{\dot{\beta}}\right\}\Phi = -2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu\Phi = 2MV_{\alpha\dot{\beta}} = 0. \tag{22}$$

The above equality means that Φ is constant in space-time and depends only on θ and $\bar{\theta}$. Taking into account the condition (19), the most general form of Φ is:

$$\Phi_c = c_1 + c_2^\alpha\theta_\alpha + \bar{c}_{3\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} + c_{4\mu}\theta\sigma^\mu\bar{\theta} + c_5(\theta\theta - \bar{\theta}\bar{\theta}) \tag{23}$$

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with constant $c_1, c_2, ..., c_5$. One further notices that the equations (17) are invariant under the simultaneous shift:

$$\begin{aligned} \Phi &\rightarrow \Phi' = \Phi + \Phi_c, \\ V^{\alpha\dot{\alpha}} &\rightarrow V'^{\alpha\dot{\alpha}} = V^{\alpha\dot{\alpha}} + V_c^{\alpha\dot{\alpha}} \end{aligned} \quad (24)$$

where

$$V_c^{\alpha\dot{\alpha}} = c_2^{\alpha}\bar{\theta}^{\dot{\alpha}} - \theta^{\alpha}\bar{c}_3^{\dot{\alpha}} + c_{4\mu}\bar{\sigma}^{\mu\dot{\alpha}\alpha}(\theta\theta - \bar{\theta}\bar{\theta}) + c_5\theta^{\alpha}\bar{\theta}^{\dot{\alpha}}. \quad (25)$$

I can perform this shift so that

$$\Phi = 0. \quad (26)$$

Eqs. (17) reduce then to:

$$\bar{D}_{\dot{\alpha}}V^{\alpha\dot{\alpha}} = D_{\alpha}V^{\alpha\dot{\alpha}} = 0. \quad (27)$$

I first express the bi-spinor superfield $V^{\alpha\dot{\alpha}}$ through the vector superfield V_μ:

$$V^{\alpha\dot{\alpha}} = \sigma^{\mu\dot{\alpha}\alpha}V_\mu \quad (28)$$

Acting with the anticomutator $\left\{D_\alpha, \bar{D}_{\dot{\beta}}\right\}$ on the superfield $V^{\alpha\dot{\alpha}}$ one notices that the superfield V_μ is divergenceless:

$$\partial^\mu V_\mu = 0. \quad (29)$$

and due to Eq.(20):

$$D^2 V_\mu = \bar{D}^2 V_\mu = 0. \quad (30)$$

To find the general solution to Eqs. (27, 29, 30) let me expand V_μ in terms of component fields:

$$\begin{aligned} V_\mu &= a_\mu(x) + \sqrt{2}\theta\psi_\mu(x) + \sqrt{2}\bar{\theta}\bar{\chi}_\mu(x) + i\theta\sigma^\nu\bar{\theta}v_{\nu\mu}(x) \quad (31) \\ &+ \theta\theta f_\mu(x) + \bar{\theta}\bar{\theta}\bar{h}_\mu(x) \\ &+ \bar{\theta}\bar{\theta}\theta^\alpha\left(\eta_{\mu\alpha}(x) - \frac{i}{\sqrt{2}}\sigma^\nu_{\alpha\dot{\alpha}}\partial_\nu\bar{\chi}_\mu^{\dot{\alpha}}(x)\right) \\ &+ \theta\theta\bar{\theta}_{\dot{\alpha}}\left(\bar{\rho}_\mu^{\dot{\alpha}}(x) + \frac{i}{\sqrt{2}}\bar{\sigma}^{\nu\dot{\alpha}\alpha}\partial_\nu\psi_{\mu\alpha}(x)\right) \\ &+ \theta\theta\bar{\theta}\bar{\theta}\left(c_\mu(x) - \frac{1}{4}\Box a_\mu(x)\right). \end{aligned}$$

The conditions (27, 29, 30) lead to the following relations among the component fields (x dependence suppressed):

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- bosons:

$$\partial^\mu a_\mu = 0, \tag{32}$$

$$f_\mu = h_\mu = c_\mu = 0, \tag{33}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^\lambda_{\alpha\dot{\beta}}(\partial_\lambda a_\mu + v_{\lambda\mu}) = 0, \tag{34}$$

$$\sigma^\lambda_{\beta\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}(\partial_\lambda a_\mu - v_{\lambda\mu}) = 0, \tag{35}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha}\Box a_\mu + \bar{\sigma}^{\mu\dot{\alpha}\beta}\sigma^\nu_{\beta\dot{\beta}}\bar{\sigma}^{\lambda\dot{\beta}\alpha}\partial_\nu v_{\lambda\mu} = 0, \tag{36}$$

$$\partial^\nu v_{\nu\mu} = \partial^\mu v_{\nu\mu} = 0. \tag{37}$$

- fermions:

$$\partial^\mu \psi_{\mu\alpha} = \partial^\mu \bar{\chi}_{\mu\dot{\alpha}} = 0, \tag{38}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi_{\mu\alpha} = \sigma^\mu_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}_\mu = 0, \tag{39}$$

$$\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta}\partial_\nu\psi_{\mu\beta} = \bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^\nu_{\alpha\dot{\beta}}\partial_\nu\bar{\chi}^{\dot{\beta}}_\mu = 0, \tag{40}$$

$$\eta^\alpha_\mu = \bar{\rho}^{\dot{\alpha}}_\mu = 0, \tag{41}$$

$$\Box\psi^\alpha_\mu = \Box\bar{\chi}^{\dot{\alpha}}_\mu = 0, \tag{42}$$

As can be seen from the above equations we can eliminate the fields f_μ, h_μ, c_μ, $\eta_{\mu\alpha}$ and $\bar{\rho}_{\mu\dot{\alpha}}$ from the supermultiplet, so that we are left with the component fields a_μ, $\psi_{\mu\alpha}$, $\bar{\chi}_{\mu\dot{\alpha}}$ and $v_{\lambda\mu}$ only. In general they form several possible supermultiplets depending on their Lorenz and spin structure, they obey also corresponding equations of motion.

3. Supermultiplets.

Systematic decomposition into irreducible supersymmetry representations can be performed by looking at the structure of the tensor field $v_{\lambda\mu}$.

3.a Symmetric case: $\quad v_{\lambda\mu} = v_{\mu\lambda}$.

Combining the relations (34) and (35) one obtains:

$$-i\epsilon^{\nu\mu\lambda\rho}v_{\lambda\rho} + \partial^\nu a^\mu - \partial^\mu a^\nu = 0. \tag{43}$$

$$-i\epsilon^{\nu\mu\lambda\rho}\partial_\lambda a_\rho + v^{\nu\mu} - v^{\mu\nu} = 0. \tag{44}$$

Symmetric field $v_{\lambda\mu}$ gives:

$$\partial_\nu a_\mu - \partial_\mu a_\nu = 0. \tag{45}$$

This means that the field a_μ must be of the form:

$$a_\mu(x) = \partial_\mu a(x). \tag{46}$$

Since the component fields are interrelated by the supersymmetry transformation:

$$\delta_\xi a_\mu \equiv (\xi Q + \bar\xi\bar Q)a_\mu = \partial_\mu(\delta_\xi a) = \sqrt{2}\xi\psi_\mu + \sqrt{2}\bar\xi\bar\chi_\mu, \tag{47}$$
$$\dots \tag{48}$$

the remaining independent fields must be also of the form:

$$\psi_\mu(x) = \partial_\mu\psi(x) \quad , \quad \bar\chi_\mu(x) = \partial_\mu\bar\chi(x), \tag{49}$$
$$v_{\lambda\mu}(x) = \partial_\lambda\partial_\mu v(x),$$
$$f_\mu(x) = \partial_\mu f(x) \quad , \quad h_\mu(x) = \partial_\mu h(x),$$
$$\dots$$

Eqs. (32-42) simplify now significantly and the "symmetric" case can be summarized: the solution to the Eqs. (17) consists of one chiral $(a+v,\ \psi_\alpha)$ and one antichiral $(a-v,\ \bar\chi_{\dot\alpha})$ supermultiplet with two scalar fields satisfying the (massless) Klein-Gordon equations and two spinor fields satisfying the (massless) Dirac equations.

$$\Box a(x) = 0 \quad , \quad \Box v(x) = 0, \tag{50}$$
$$\bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu\psi_\alpha(x) = 0 \quad , \quad \sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar\chi^{\dot\alpha}(x) = 0,$$

All the remaining component fields vanish.

3.b Antisymmetric case: $\quad v_{\lambda\mu} = -v_{\mu\lambda}$.

In this case I can express $v^{\mu\nu}$ through a vector field v^μ:

$$v^{\mu\nu} = \partial^\mu v^\nu - \partial^\nu v^\mu. \tag{51}$$

and relate it to the tensor field

$$a^{\mu\nu} = \partial^\mu a^\nu - \partial^\nu a^\mu \tag{52}$$

through (imaginary) duality relation:

$$v_{\nu\mu} = \frac{i}{4}\epsilon_{\nu\mu\lambda\rho}a^{\lambda\rho}. \qquad (53)$$

The relations (32-37) can be expressed by one of these fields (eg. $a^{\mu\nu}$) and lead to the Maxwell field equations:

$$\partial^\nu a_{\nu\mu} = 0, \quad \epsilon_{\nu\mu\lambda\rho}\partial^\mu a^{\lambda\rho} = 0. \qquad (54)$$

with the Lorenz condition $\partial^\mu a_\mu = 0$.

Let us look now at the fermionic sector. The component fields η_μ^α and $\bar{\rho}_\mu^{\dot\alpha}$ vanish. The vector-spin fields obey Eqs (38,39) which means that only the spin 3/2 is present. In addition they obey the Eqs. (40) which can be rewritten in the form of the Rarita-Schwinger field equations:

$$\epsilon_{\rho\mu\lambda\nu}\bar{\sigma}^{\mu\dot\alpha\alpha}\partial^\lambda\psi_\alpha^\nu = 0, \quad \epsilon_{\rho\mu\lambda\nu}\sigma_{\alpha\dot\alpha}^\mu\partial^\lambda\bar{\chi}^{\nu\dot\alpha} = 0. \qquad (55)$$

To summarize the antisymmetric case: the solution to Eqs.(17) is built of two spin 3/2 vector-spinor component fields $\psi_\mu^\alpha(x)$ and $\bar{\chi}_\mu^{\dot\alpha}(x)$ obeying the (massless) Rarita-Schwinger equations and a complex spin 1 vector field $a_\mu(x)$ obeying the Maxwell equations. The above multiplets are sometimes called matter gravitino supermultiplets [5].

4. Summary.

Together with earlier results [4], the "square root" of the Dirac operator $\mathcal{S}$, when acting on the superfields F and B, gives all superfields necessary for the construction of renormalizable supersymmetric theories together with the appropriate equations of motion.

Two issues are certainly worth further study. The first concerns the masses. Even if we started from the massive Dirac equation, all resulting component fields are massless. This seems to be quite natural in supersymmetry at classical level.

The gauge invariant interaction may be introduced in the way suggested in Ref. [3] where the spinorial derivatives $D_\alpha, \bar{D}_{\dot\alpha}$ appearing in the operator $\mathcal{S}$ are replaced by the covariant spinorial derivatives $\mathcal{D}_\alpha, \mathcal{D}_{\dot\alpha}$:

$$\mathcal{D}_\alpha = D_\alpha + igA_\alpha, \quad \mathcal{D}_{\dot\alpha} = \bar{D}_{\dot\alpha} + ig\bar{A}_{\dot\alpha}. \qquad (56)$$

The above problems are currently under investigation.

5. Acknowledgements.

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